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COMPUTATION OF REGULARIZATION PARAMETERS USING THE FOURIER COEFFICIENTS

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Abstract

In the solution of ill-posed problems by means of regularization methods, a crucial issue is the computation of the regularization parameter. In this work, we focus on the Truncated Singular Value Decomposition (TSVD) and Tikhonov method, and we define a method for computing the regularization parameter based on the behavior of Fourier coefficients. We compute a safe index for truncating the TSVD and consequently a value for the regularization parameter of the Tikhonov method. An extensive numerical experimentation is carried out on the Hansen's Regtool [2] test problems, and the results confirm the effectiveness and robustness of the method proposed.

1. Introduction

In this work, discrete ill-posed problems are solved by means of regularization methods based on Singular Value Decomposition (SVD). By discrete ill-posed problems, we intend a class of least squares problems:

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$$\min_{x} \|Hf - g\|_2, \quad H \in \mathbb{R}^{m \times n},\ g \in \mathbb{R}^m, \quad m \geq n,$$

where the matrix H is ill conditioned with smoothly decaying singular values. Such matrices are obtained by the discretization of ill-posed problems such as Fredholm first kind integral equations that model many imaging problems. These problems are very sensitive to small data perturbations such as noise present in the data g, usually represented by a random process, or errors in the matrix H.

A great variety of direct and iterative regularization methods can be found in the literature, they mainly replace the ill-posed problem with a nearby well posed one, which is less sensitive to perturbations. One well known method is the Tikhonov method, which computes a regularized solution x_λ as the minimizer of the discrete Tikhonov functional:

$$\min_{x} \|Hf - g\|_2^2 + \lambda \|Lf\|_2^2,$$

where $\lambda > 0$ is the regularization parameter and L is either the identity matrix or a well conditioned discrete approximation to some derivative operator.

For practical implementation of such methods, it is necessary to compute a suitable value of the regularization parameter λ.

In this work, we focus on small-medium size problems, where the Singular Value Decomposition (SVD) can be efficiently computed. Analyzing the behavior of the Fourier coefficients in terms of the discrete Picard condition [1], we define a rule for computing the index for truncating the singular values in the Truncated Singular Value Decomposition (TSVD) method, and consequently a value for the regularization parameter λ of the Tikhonov method.

The effectiveness and robustness of our rule is evaluated by comparison to one of the most widely used and successful method, i.e., the Generalized Cross Validation method.

In Section 2, the properties of the TSVD and Tikhonov methods are reported with respect to the value of the regularization parameter. In Section 3, the rule proposed is reported and analyzed. Finally, in Section

4, an extensive numerical experimentation is carried out on the Hansen's Regtool [2] test problems.

2. The Discrete Picard Condition

In this section, we recall the main properties of the regularization methods with respect to the SVD.

Let f_{LS} be the minimal norm solution of the least squares problem:

$$\min_f \|Hf - g\|^2, \quad H \in \mathbb{C}^{m\times n},\ g \in \mathbb{C}^m,\ (m \geq n). \tag{1}$$

Let r be the rank of the matrix H and let $H = USV^*$ be the SVD of H. It is possible to characterize f_{LS} in terms of the SVD of the matrix H:

$$f_{LS} = \sum_{i=1}^{n} \frac{u_i^* g}{\sigma_i} v_i,$$

where σ_i are the singular values, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$, $u_i (i = 1, \dots, m)$, and $v_i (i = 1, \dots, n)$ are the columns of the unitary matrices U and V, respectively. When (1) is the discretization of an ill-posed problem (such as the Fredholm first kind integral equation), f_{LS} is dominated by the errors present in g (data noise) and H, therefore, it is necessary to introduce regularization methods for filtering out the components that cause the errors. A regularized solution can be computed as follows:

$$f_{reg} = \sum_{i=1}^{n} \Phi(\sigma_i) \frac{u_i^* g}{\sigma_i} v_i, \tag{2}$$

where Φ is the *filter function* with values in the interval $[0, 1]$. By means of the *filter function*, we characterize the following regularization methods:

- Truncated Singular Value Decomposition method (TSVD), where $\Phi \equiv \Phi_k$ is an ideal low pass filter:

$$\Phi_k(\sigma_i) = \begin{cases} 1, & \text{if } \sigma_i >= \sigma_k (\text{i.e., } i <= k), \\ 0, & \text{otherwise,} \end{cases}$$

where the threshold value σ_k is defined by the *regularization parameter* k.

• Tikhonov regularization method (Tikh), where $\Phi \equiv \Phi_\alpha$ is a least squares smoothing filter:

$$\Phi_\alpha(\sigma_i) = \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2},$$

and the value of the *regularization parameter* α defines the behavior of the filter function as follows:

$-\alpha << \sigma_i \Rightarrow \Phi_\alpha(\sigma_i) \approx 1$, i.e., the i-th term in (2) is almost unchanged;

$-\alpha >> \sigma_i \Rightarrow \Phi_\alpha(\sigma_i) \approx 0$, i.e., the i-th term in (2) is damped to zero.

The function Φ_α behaves like a smooth low pass filter of the solution terms, damping the components relative to the singular values smaller than α.

In both cases, the proper determination of the regularization parameter is a crucial issue: too small values of α or σ_k (i.e., large values of k in TSVD) produce a noisy solution dominated by the errors present in the system, while large values of α (i.e., small values of k in TSVD) produce a smooth blurred solution.

The decay rate of the Fourier coefficients $|u_i^* g|$ with respect to the singular values σ_i is a key point to determine the properties of the regularized solution f_{reg}.

The discrete Picard condition [1] is a property of the ratio between the Fourier coefficients $|u_i^* g|$ and the singular values σ_i, which is stated as follows.

If the Fourier coefficients $|u_i^* g|$ *decay to zero faster than the singular values, then the regularized solution* f_{reg} *has the same regularity properties as the exact solution f.*

The check of this property may not be easy since $|u_i^* g|$ may have a non monotonic behavior. A rule based on the moving geometric mean has been proposed in [1]. Further, analysis of the Fourier coefficients concerns the terms that can be included in the regularized solution. As pointed out in [3] (pg. 70, 71):

The number of terms k' *that can be safely included in the solution* f_{reg} *is such that:*

$$k' \leq \min(i_H, i_g), \tag{3}$$

where i_H *is the index at which* σ_i *begin to level off and* i_g *is the index at which* $|u_i^* g|$ *begin to level off. The value* i_H *is proportional to the error present in the matrix H* (*i.e.*, *model error*), *while the value* i_g *is proportional to the errors present in the data g* (*i.e.*, *noise error*).

An example of this issue is shown in Figure 1 reporting the plots of the singular values σ_i (blue line), $|u_i^* g|$ (green crosses), and $|u_i^* g| / \sigma_i$ (red o) for the Foxgood (1(a), 1(b)), and Phillips (1(c), 1(d)) test problems taken from Hansens's Regtool, both in absence and presence of noise on the data g (see Section 4 for the details).

We observe that in the Foxgood test problem $25 \leq i_H \leq 35$, while $i_g \simeq 30$ for noiseless case and $i_g < 10$ at high noise values. For this test problem, the amount of noise present in the data determines the value k' in (3) and we can observe a quite evident minimum value in the curve $|u_i^* g| / \sigma_i$.

In the Phillips test problem, we can observe that the singular values do not level off at machine epsilon $(\sim 2.2e^{-16})$, but have a smooth decreasing behavior toward a minimum value $> 10^{-5}$, so we have

$i_H \simeq n$, where n is the dimension of the problem ($n = 100$ in this example). In the noiseless case, the coefficients $|u_i^* g|$ can be splitted into two successions: one that level off at machine epsilon (we can interpret this as numerically zero values), and the other that decreases faster than the singular values curve and fulfils the Picard condition. In this case, the value of k' in (3) is given by i_H and practically no truncation is needed in the TSVD (Figure 1(c)). In the case of high noise, we observe that the values $|u_i^* g|$ settle down around the noise value corresponding to increasing values of the curve $|u_i^* g| / \sigma_i$ (Figure 1(d)).

If the value at which the singular values level off is very small (machine epsilon), then the index k' is determined by the value of i_g (error data), otherwise, the model error is dominant in the low noise cases.

3. The MinMax Rule

We define here a criterion for computing the values of the regularization parameter based on the terms that can be safely included in the f_{reg} solution.

Definition 1 (MinMax Rule)**.** Let φ_i be the succession of the solution coefficients

$$\varphi_i = \frac{|u_i^* g|}{\sigma_i}, \quad i = 1, \dots, n,$$

and let us separate the terms φ_i in two sets:

$$\rho_1 = \{\varphi_i : \varphi_i \geq \sigma_i\}, \quad \rho_2 = \{\varphi_i : \varphi_i < \sigma_i\}.$$

Let K_1 be the number of elements in ρ_1, K_2 be the number of elements in ρ_2 and $S_{\min} = \min_i(\sigma_i)$, compute the index k^* such that:

Case I. If $S_{\min} \leq 1.e - 13$ or $K_2 = 0$ (i.e., the sequence ρ_2 is empty), then k^* is the index relative to the minimum value in the sequence ρ_1.

Case II. Otherwise, k^* is the smallest index among the last 5 elements in ρ_2 such that

$$\frac{|u_{k^*+1}^* g|}{\sigma_{k^*+1}} > \sigma_{k^*+1}.$$

Then k^* is chosen as the regularization parameter of the TSVD method, and

$$\alpha^* = \sigma(k^*)$$

is the regularization parameter for the Tikhonov method.

When the minimum singular value $(S_{\min})$ is close to the machine epsilon (or the sequence ρ_2 is empty), then ρ_1 is used to determine the regularization index. In this case, the data noise is assumed to be predominant with respect to the model errors and the sequence ρ_1 shows a quite evident minimum value.

An example of this case is reported in Figure 2 relative to the Foxgood test problem. In this case, only the sequence ρ_1 is computed since $K_2 = 0$ as shown in Figure 2(a) for the noiseless case, and in Figure 2(c) for the high noise case. In the noiseless case, ρ_1 has a spurious minimum (Figure 2(a)) that causes the error to be greater to the optimal one (2(b)). In the noisy case, (Figure 2(c)), we observe that the computed value is very close to the optimal (Figure 2(d)).

In the Case II of the MinMax Rule, the model errors are predominant and the greatest indices in ρ_2 are included in the TSVD provided that they are contiguous (i.e., the successive element does not belong to ρ_1). This situation is shown in Figure 3 obtained with the Phillips test problem. In the noiseless case, the computed index is not close to the optimal one (Figure 3(a)), but we can see that the relative error is constant so the solution is effectively as good as the optimal one (3(b)). In the noisy case, we observe that the computed value is very close to the optimal (Figures 3(c), 3(d)).

4. Numerical Experiments

In this section, we report the results obtained by several test problems with the MinMax Rule and the Generalized Cross Validation method implemented in Hansen's Regtool (GCV function) [2].

The test is carried out on the 17 Regtools test problems relative to $1D$ integral equations. We subdivide the test problems into two groups named $G1$ and $G2$, respectively. The group $G1$ is constituted by the test problems with low errors present in the matrix (i.e., with singular values that level-off at machine epsilon or lower values). Table (1) reports the names of the test and the call to the specific matlab function used to generate the test. The group $G2$ is constituted by the test problems with quite large errors in the matrix (i.e., with singular values decay to a minimum value much larger than machine epsilon). The names of the test problems and function calls are reported in Table (2).

Each problem defines the matrix H, the true solution f_{true} and the *r.h.s.* g of the least squares problem $\min\|Hf_{true} - g\|$. White gaussian noise is added to the data g with variance δ and the problem $\min\|Hf - g^{\delta}\|$ is solved by means of the TSVD and Tikhonov methods. The noisy data g^{δ} are obtained as follows:

$$g^{\delta} = g + \delta\|g\|\underline{\eta},$$

where $\underline{\eta}$ is a unitary vector of gaussian distributed random values. All the tests are performed with dimension $n = 100$.

In Tables 3, 4, and 7 are reported the results obtained by the TSVD method: the first two columns are relative to the test problems and report the name of the test function (**Test** column) and the value of δ (**Noise** column). The results reported in columns **Best** are obtained by computing the optimal solution: the column *Index* is the number of terms used in the TSVD solution (2), and the column *Error* reports the relative error computed with respect to the solution f_{true}.

The solution obtained by MinMax Rule is reported in columns **MinMax_tsvd**, where the columns *Index* and *Error* report the number of terms determined by the MinMax Rule and the relative error, respectively. Analogously, the columns **GCV_tsvd** report the number of terms (column *Index*) and the relative error (column *Error*) computed by the GCV function.

Observing the last column of Tables 3, 4, 5, and 6 (relative to the test problems $G1$), we can see that GCV presents several critical situations with errors greater than 10, while MinMax Rule has always maximum error < 1.

This behavior is not present in Tables 7 and 8 showing that test problems like $G2$ are better than $G1$ for GCV method. Analyzing the errors, we can see some cases, where MinMax Rule and GCV have maximum error in the interval [1, 6] for the TSVD method (Table 7), while the GCV has better performance for the Tikhonov method (Table 8).

The efficiency of each method Ef can be evaluated as the ratio between the optimal error E_b (i.e., the relative error obtained using the optimal regularization parameter), and the error computed by each method E_m:

$$Ef = \frac{E_b}{E_m}.$$

The average behavior of each method with respect to the sets of test, problems $G1$ and $G2$ is analyzed by computing the smallest efficiency value $Ef_{\min}$ and the mean efficiency value Ef_{mean} as reported in Table 9. Moreover, the same table reports the percentages of tests, where each method reached the maximum efficiency value $(Ef = 1)$ (column *Opt*%), and those where the method reported a relative error greater than 10 (column *Fail*%).

The MinMax Rule performs better than GCV in terms of average efficiency for the tsvd method with both tests $G1$ and $G2$ and for the Tikhonov method with test $G1$. The percentage of fails reported in these cases are 29 and 13 showing that GCV is not always able to compute an acceptable value for the regularization parameter.

For test problems $G2$ with the Tikhonov method, we can see that GCV has better efficiency values with respect to the MinMax Rule. Both methods are, however, robust for these test problems since they do not report failures.

The minimum efficiency value reported in Table 9 is relative to the test ilaplace_ex1 in absence of noise, solved with GCV_tsvd method. The solution and errors are reported in Figure 4 show very bad reconstructions, the small value in the efficiency parameter is due to a very large value of the relative error obtained by the regularization method (Figure 4(b)), and the computed regularization parameter is larger than the optimal.

The minimum efficiency value obtained by the MinMax Rule is relative to the case test heat in absence of noise, solved tsvd method. In this case, we can observe a good quality solution (Figure 5(a)) since the small efficiency value is due to a very small value of the relative error in the optimal solution, and the value of the regularization parameter is smaller than the optimal.

We can therefore conclude that GCV is a very efficient method for computing the Tikhonov regularization parameter in test problems with predominant the errors in the matrix ($G2$ group) returning in many cases values very close to the optimal. In all the remaining cases, MinMax Rule is more robust, preventing failures, and giving always good results.

5. Conclusions

A fast and robust method for computing the regularization parameter is proposed. The method is based on the properties of the Fourier coefficients obtained by computing the SVD. This strategy can be extended quite naturally to general regularization problems of the kind:

$$\min_f \|Hf - g\|^2 + \|Lf\|^2, \quad L \neq I,$$

by means of the Generalized Singular Value Decomposition (GSVD).

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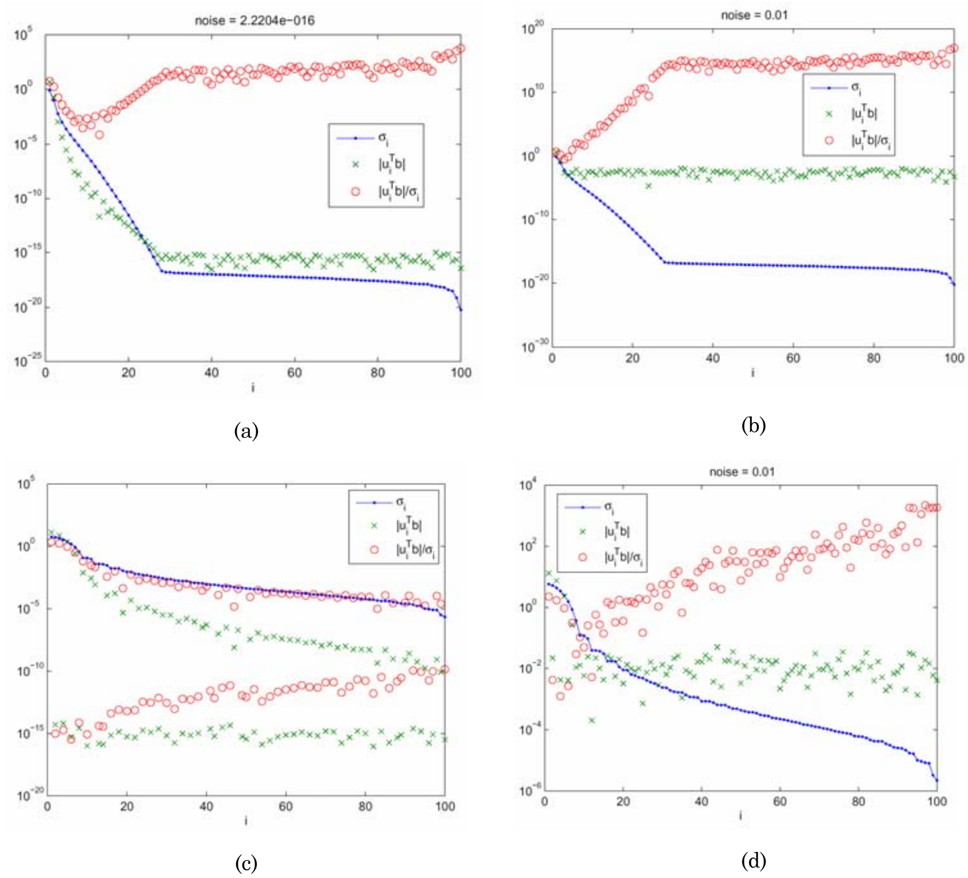


Figure 1. (a)(b) Foxgood test problem: Coefficients, singular values and Fourier coefficients: (a) noiseless case, (b) high noise =1.*e*–2; (c), (d) Phillips test problem: Coefficients, singular values and Fourier coefficients: (c) noiseless case, (d) high noise = 1.*e*–2.

Table 1. Group $G1$ test problems: $n = 100$

Test	Regtool function
Shaw	shaw(n)
Heat	heat(n)
Baart	baart(n)
iLaplace_ex1	i_laplace(n, 1)
iLaplace_ex2	i_laplace(n, 2)
iLaplace_ex3	i_laplace(n, 3)
Foxgood	foxgood(n)
Wing	wing(n)
Spikes	spikes(n)
Gravity_ex1	gravity(n, 1)
Gravity_ex2	gravity(n, 2)
Gravity_ex3	gravity(n, 3)

Table 2. Group $G2$ test problems: $n = 100$

Test	Regtool function
Phillips	phillips(n)
Deriv2_ex1	deriv2(n, 1)
Deriv2_ex2	deriv2(n, 2)
Deriv2_ex3	deriv2(n, 3)

Table 3. TSVD method with $G1$ test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tsvd** regularization parameter and relative error of MinMax Rule. Columns **GCV_tsvd** regularization parameter and relative error of GCV function

Test	Noise	Best		MinMax_tsvd		GCV_tsvd	
		Index	*Error*	*Index*	*Error*	*Index*	*Error*
Shaw	0.0*e*+000	19	3.53*e*–004	14	3.65*e*–003	45	**1.58*e*+001**
Shaw	1.0*e*–005	10	2.72*e*–002	8	4.72*e*–002	9	3.21*e*–002
Shaw	1.0*e*–003	7	4.87*e*–002	8	8.81*e*–002	99	**5.38*e*+014**
Shaw	1.0*e*–002	7	1.03*e*–001	5	1.47*e*–001	7	1.03*e*–001
Heath	0.0*e*+000	97	3.04*e*–011	56	1.04*e*–002	99	**5.42*e*+006**
Heath	1.0*e*–005	55	1.14*e*–002	50	1.35*e*–002	58	1.29*e*–002
Heath	1.0*e*–003	24	4.39*e*–002	30	4.58*e*–002	29	4.56*e*–002
Heath	1.0*e*–002	23	1.19*e*–001	21	1.27*e*–001	17	1.31*e*–001
Baart	0.0*e*+000	11	1.63*e*–002	10	1.93*e*–002	64	4.81*e*+000
Baart	1.0*e*–005	5	5.25*e*–002	5	5.25*e*–002	5	5.25*e*–002
Baart	1.0*e*–003	5	9.26*e*–002	5	9.26*e*–002	7	**1.58*e*+003**
Baart	1.0*e*–002	3	1.67*e*–001	3	1.67*e*–001	3	1.67*e*–001
Foxgood	0.0*e*+000	7	2.47*e*–004	13	8.27*e*–004	38	**3.66*e*+001**
Foxgood	1.0*e*–005	5	1.56*e*–003	5	1.56*e*–003	4	2.24*e*–003
Foxgood	1.0*e*–003	3	7.86*e*–003	3	7.86*e*–003	15	**8.55*e*+004**
Foxgood	1.0*e*–002	2	3.33*e*–002	3	9.20*e*–002	2	3.33*e*–002
Wing	0.0*e*+000	7	3.11*e*–001	3	5.95*e*–001	10	4.62*e*+000
Wing	1.0*e*–005	5	4.24*e*–001	3	5.95*e*–001	4	4.38*e*–001
Wing	1.0*e*–003	2	5.95*e*–001	3	5.95*e*–001	2	5.95*e*–001
Wing	1.0*e*–002	2	5.95*e*–001	3	6.30*e*–001	2	5.95*e*–001
Spikes	0.0*e*+000	21	3.70*e*–001	9	8.38*e*–001	27	7.78*e*+000
Spikes	1.0*e*–005	12	7.79*e*–001	9	8.38*e*–001	12	7.79*e*–001
Spikes	1.0*e*–003	10	8.15*e*–001	7	8.56*e*–001	99	**3.56*e*+014**
Spikes	1.0*e*–002	8	8.39*e*–001	9	8.41*e*–001	6	8.56*e*–001

Table 4. TSVD method with $G1$ test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tsvd** regularization parameter and relative error of MinMax Rule. Columns **GCV_tsvd** regularization parameter and relative error of GCV function

Test	Noise	Best		MinMax_tsvd		GCV_tsvd	
		Index	*Error*	*Index*	*Error*	*Index*	*Error*
iLaplace_ex1	0.0*e*+000	25	4.07*e*–006	26	4.45*e*–006	98	**1.45*e*+014**
iLaplace_ex1	1.0*e*–005	13	4.32*e*–002	14	5.04*e*–002	13	4.32*e*–002
iLaplace_ex1	1.0*e*–003	9	1.08*e*–001	7	1.27*e*–001	16	1.20*e*+002
iLaplace_ex1	1.0*e*–002	6	1.48*e*–001	7	1.73*e*–001	6	1.48*e*–001
iLaplace_ex2	0.0*e*+000	31	7.14*e*–001	31	7.14*e*–001	30	7.16*e*–001
iLaplace_ex2	1.0*e*–005	16	7.69*e*–001	15	7.71*e*–001	14	7.70*e*–001
iLaplace_ex2	1.0*e*–003	10	7.95*e*–001	11	8.08*e*–001	16	**6.81*e*+001**
iLaplace_ex2	1.0*e*–002	8	8.08*e*–001	7	8.15*e*–001	7	8.15*e*–001
iLaplace_ex3	0.0*e*+000	24	1.40*e*–007	24	1.40*e*–007	23	2.42*e*–007
iLaplace_ex3	1.0*e*–005	11	2.71*e*–003	11	2.71*e*–003	10	4.57*e*–003
iLaplace_ex3	1.0*e*–003	8	2.00*e*–002	8	2.00*e*–002	16	1.37*e*+002
iLaplace_ex3	1.0*e*–002	5	7.65*e*–002	8	1.56*e*–001	5	7.65*e*–002
iLaplace_ex4	0.0*e*+000	25	7.29*e*–001	25	7.29*e*–001	37	**4.01*e*+002**
iLaplace_ex4	1.0*e*–005	16	7.65*e*–001	15	7.67*e*–001	14	7.66*e*–001
iLaplace_ex4	1.0*e*–003	10	7.91*e*–001	11	8.03*e*–001	16	**6.49*e*+001**
iLaplace_ex4	1.0*e*–002	8	8.06*e*–001	8	8.06*e*–001	7	8.10*e*–001
Gravity_ex1	0.0*e*+000	35	1.20*e*–006	36	1.59*e*–006	96	**6.40*e*+001**
Gravity_ex1	1.0*e*–005	14	2.72*e*–003	15	3.91*e*–003	13	3.53*e*–003
Gravity_ex1	1.0*e*–003	10	1.65*e*–002	10	1.65*e*–002	8	2.03*e*–002
Gravity_ex1	1.0*e*–002	7	3.00*e*–002	8	4.42*e*–002	6	4.13*e*–002
Gravity_ex2	0.0*e*+000	45	2.19*e*–003	27	4.39*e*–003	43	2.33*e*–003
Gravity_ex2	1.0*e*–005	16	1.01*e*–002	14	1.03*e*–002	13	1.04*e*–002
Gravity_ex2	1.0*e*–003	9	2.39*e*–002	10	2.46*e*–002	16	3.17*e*+000
Gravity_ex2	1.0*e*–002	7	5.02*e*–002	10	1.83*e*–001	9	1.80*e*–001
Gravity_ex3	0.0*e*+000	48	3.14*e*–002	36	3.79*e*–002	47	3.16*e*–002
Gravity_ex3	1.0*e*–005	17	5.79*e*–002	16	5.98*e*–002	15	5.99*e*–002
Gravity_ex3	1.0*e*–003	9	7.50*e*–002	10	7.52*e*–002	16	2.97*e*+000
Gravity_ex3	1.0*e*–002	8	9.78*e*–002	10	1.86*e*–001	8	9.78*e*–002

Table 5. Tikhonov method with $G1$ test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tikh** regularization parameter and relative error of MinMax Rule. Columns **GCV_tikh** regularization parameter and relative error of GCV function

Test	**Noise**	**Best**		**MinMax_tikh**		**GCV_tikh**	
		Value	*Error*	*Value*	*Error*	*Value*	*Error*
Shaw	0.0e+000	5.44e–011	5.33e–004	6.43e–008	3.62e–003	1.12e–012	3.17e–004
Shaw	1.0e–005	4.34e–003	4.55e–002	4.34e–003	4.55e–002	1.79e–004	3.12e–002
Shaw	1.0e–003	4.34e–003	4.86e–002	4.34e–003	4.86e–002	1.22e–003	1.41e–001
Shaw	1.0e–002	5.90e–002	1.19e–001	5.90e–002	1.19e–001	1.94e–002	6.32e–002
Heath	0.0e+000	1.25e–006	1.59e–003	7.82e–005	8.89e–003	3.49e–014	3.06e–011
Heath	1.0e–005	1.01e–004	1.01e–002	1.32e–004	1.08e–002	4.16e–005	1.29e–002
Heath	1.0e–003	1.34e–003	3.89e–002	9.98e–004	4.27e–002	7.72e–004	5.14e–002
Heath	1.0e–002	3.10e–003	1.11e–001	3.10e–003	1.11e–001	2.75e–003	1.21e–001
Baart	0.0e+000	4.56e–012	1.94e–002	4.56e–012	1.94e–002	1.23e–014	3.37e–001
Baart	1.0e–005	2.36e–004	7.41e–002	2.36e–004	7.41e–002	3.63e–005	5.66e–002
Baart	1.0e–003	2.36e–004	1.46e–001	2.36e–004	1.46e–001	1.72e–007	**1.20e+003**
Baart	1.0e–002	7.16e–002	2.17e–001	7.16e–002	2.17e–001	1.17e–002	1.31e–001
Foxgood	0.0e+000	2.77e–006	3.14e–004	2.72e–008	7.92e–004	3.08e–015	2.61e–001
Foxgood	1.0e–005	2.57e–004	2.42e–003	2.57e–004	2.42e–003	2.19e–004	2.62e–003
Foxgood	1.0e–003	6.60e–003	1.44e–002	6.60e–003	1.44e–002	1.09e–009	**7.75e+004**
Foxgood	1.0e–002	6.60e–003	6.19e–002	6.60e–003	6.19e–002	1.00e–002	4.97e–002
Wing	0.0e+000	3.71e–007	3.55e–001	1.11e–003	5.95e–001	1.70e–015	3.14e–001
Wing	1.0e–005	1.11e–003	5.95e–001	1.11e–003	5.95e–001	1.84e–006	4.37e–001
Wing	1.0e–003	1.11e–003	5.94e–001	1.11e–003	5.94e–001	3.52e–011	**5.79e+005**
Wing	1.0e–002	1.11e–003	6.05e–001	1.11e–003	6.05e–001	2.00e–003	5.97e–001
Spikes	0.0e+000	4.41e–012	6.17e–001	1.02e–002	8.35e–001	6.82e–014	5.00e–001
Spikes	1.0e–005	2.42e–003	8.15e–001	1.02e–002	8.35e–001	2.72e–005	7.86e–001
Spikes	1.0e–003	3.75e–002	8.42e–001	1.22e–001	8.53e–001	2.10e–003	8.05e–001
Spikes	1.0e–002	1.02e–002	8.60e–001	1.02e–002	8.60e–001	7.06e–002	8.51e–001

Table 6. Tikhonov method with $G1$ test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tikh** regularization parameter and relative error of MinMax Rule. Columns **GCV_tikh** regularization parameter and relative error of GCV function

Test	**Noise**	**Best**		**MinMax_tikh**		**GCV_tikh**	
		Value	*Error*	*Value*	*Error*	*Value*	*Error*
iLaplace_ex1	0.0e+000	3.52e–012	5.76e–006	3.52e–012	5.76e–006	7.80e–012	4.87e–006
iLaplace_ex1	1.0e–005	2.76e–005	5.30e–002	2.76e–005	5.30e–002	6.16e–005	4.64e–002
iLaplace_ex1	1.0e–003	2.51e–002	1.27e–001	2.51e–002	1.27e–001	2.18e–006	**8.68e+001**
iLaplace_ex1	1.0e–002	2.51e–002	1.50e–001	2.51e–002	1.50e–001	3.04e–002	1.52e–001
iLaplace_ex2	0.0e+000	1.87e–017	1.54e+000	1.87e–017	1.54e+000	9.02e–015	7.20e–001
iLaplace_ex2	1.0e–005	9.57e–006	7.70e–001	9.57e–006	7.70e–001	1.62e–005	7.70e–001
iLaplace_ex2	1.0e–003	5.89e–004	7.94e–001	5.89e–004	7.94e–001	2.18e–006	**4.91e+001**
iLaplace_ex2	1.0e–002	2.51e–002	8.15e–001	2.51e–002	8.15e–001	1.25e–002	8.07e–001
iLaplace_ex3	0.0e+000	9.58e–011	1.31e–007	9.58e–011	1.31e–007	2.05e–012	1.56e–005
iLaplace_ex3	1.0e–005	5.89e–004	4.20e–003	5.89e–004	4.20e–003	4.58e–004	4.44e–003
iLaplace_ex3	1.0e–003	1.02e–002	3.16e–002	1.02e–002	3.16e–002	2.18e–006	**9.90e+001**
iLaplace_ex3	1.0e–002	2.51e–002	1.11e–001	1.02e–002	1.62e–001	2.68e–002	1.09e–001
iLaplace_ex4	0.0e+000	1.91e–011	7.29e–001	1.91e–011	7.29e–001	9.02e–015	2.06e+000
iLaplace_ex4	1.0e–005	9.57e–006	7.66e–001	9.57e–006	7.66e–001	1.37e–005	7.66e–001
iLaplace_ex4	1.0e–003	5.89e–004	7.93e–001	5.89e–004	7.93e–001	2.18e–006	**4.68e+001**
iLaplace_ex4	1.0e–002	1.02e–002	8.02e–001	1.02e–002	8.02e–001	1.06e–002	8.03e–001
Gravity_ex1	0.0e+000	2.47e–010	1.41e–006	1.22e–010	1.54e–006	5.08e–010	1.36e–006
Gravity_ex1	1.0e–005	1.07e–003	3.71e–003	1.07e–003	3.71e–003	2.02e–003	3.11e–003
Gravity_ex1	1.0e–003	3.03e–002	1.92e–002	3.03e–002	1.92e–002	4.20e–004	2.03e+000
Gravity_ex1	1.0e–002	1.12e–001	4.22e–002	1.12e–001	4.22e–002	1.18e–001	4.05e–002
Gravity_ex2	0.0e+000	3.45e–012	2.34e–003	2.77e–007	4.17e–003	1.24e–012	2.27e–003
Gravity_ex2	1.0e–005	2.10e–003	1.04e–002	2.10e–003	1.04e–002	1.88e–003	1.04e–002
Gravity_ex2	1.0e–003	3.03e–002	2.16e–002	3.03e–002	2.16e–002	4.20e–004	2.17e+000
Gravity_ex2	1.0e–002	5.86e–002	7.82e–002	3.03e–002	1.54e–001	8.33e–002	4.30e–002
Gravity_ex3	0.0e+000	4.00e–013	3.20e–002	5.02e–010	3.75e–002	1.01e–013	3.16e–002
Gravity_ex3	1.0e–005	5.41e–004	5.83e–002	5.41e–004	5.83e–002	4.57e–004	5.80e–002
Gravity_ex3	1.0e–003	3.03e–002	7.14e–002	3.03e–002	7.14e–002	4.20e–004	2.04e+000
Gravity_ex3	1.0e–002	3.03e–002	1.75e–001	3.03e–002	1.75e–001	1.11e–001	1.05e–001

Table 7. TSVD method with $G2$ test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tsvd** regularization parameter and relative error of MinMax Rule. Columns **GCV_tsvd** regularization parameter and relative error of GCV function

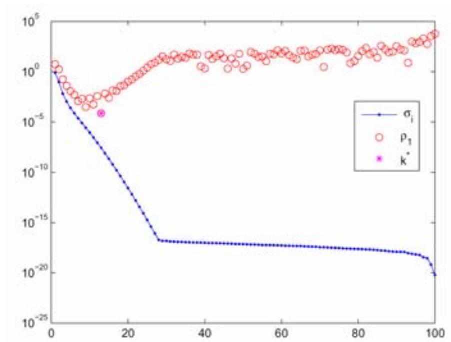
Test	Noise	Best		MinMax_tsvd		GCV_tsvd	
		Index	*Error*	*Index*	*Error*	*Index*	*Error*
Phillips	0.0e+000	99	7.57e–004	84	7.57e–004	99	7.57e–004
Phillips	1.0e–005	21	1.67e–003	24	2.24e–003	21	1.67e–003
Phillips	1.0e–003	11	1.22e–002	11	1.22e–002	13	3.25e–002
Phillips	1.0e–002	9	2.17e–002	10	2.71e–002	7	2.50e–002
Deriv2_ex1	0.0e+000	100	3.43e–012	100	3.43e–012	73	6.50e–002
Deriv2_ex1	1.0e–005	100	4.84e–002	86	5.80e–002	62	8.07e–002
Deriv2_ex1	1.0e–003	25	1.81e–001	23	1.84e–001	99	4.46e+000
Deriv2_ex1	1.0e–002	11	2.47e–001	26	1.04e+000	12	3.26e–001
Deriv2_ex2	0.0e+000	100	8.29e–006	100	8.29e–006	73	6.04e–002
Deriv2_ex2	1.0e–005	97	5.21e–002	64	7.43e–002	61	7.66e–002
Deriv2_ex2	1.0e–003	25	1.78e–001	32	2.52e–001	99	4.83e+000
Deriv2_ex2	1.0e–002	11	2.33e–001	32	1.61e+000	12	3.34e–001
Deriv2_ex3	0.0e+000	99	5.37e–004	86	5.38e–004	99	5.37e–004
Deriv2_ex3	1.0e–005	33	2.72e–003	30	2.92e–003	29	2.91e–003
Deriv2_ex3	1.0e–003	7	1.86e–002	2	1.20e–001	99	5.63e+000
Deriv2_ex3	1.0e–002	5	3.88e–002	2	1.20e–001	3	4.87e–002

Table 8. Tikhonov method with $G2$ test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tikh** regularization parameter and relative error of MinMax Rule. Columns **GCV_tikh** regularization parameter and relative error of GCV function

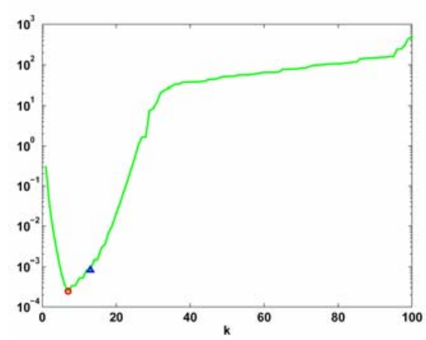
Test	Noise	Best		MinMax_tikh		GCV_tikh	
		Value	*Error*	*Value*	*Error*	*Value*	*Error*
Phillips	0.0e+000	4.26e–005	7.56e–004	4.26e–005	7.56e–004	2.26e–006	7.57e–004
Phillips	1.0e–005	5.92e–003	2.04e–003	5.13e–003	2.20e–003	4.07e–003	2.63e–003
Phillips	1.0e–003	9.48e–002	1.23e–002	9.48e–002	1.23e–002	2.71e–002	3.60e–002
Phillips	1.0e–002	1.18e–001	5.31e–002	1.18e–001	5.31e–002	1.18e–001	5.32e–002
Deriv2_ex1	0.0e+000	8.33e–006	3.00e–002	8.33e–006	3.00e–002	8.49e–006	3.07e–002
Deriv2_ex1	1.0e–005	8.36e–006	4.45e–002	9.58e–006	4.64e–002	8.49e–006	4.47e–002
Deriv2_ex1	1.0e–003	1.83e–004	1.76e–001	1.83e–004	1.76e–001	1.55e–004	1.87e–001
Deriv2_ex1	1.0e–002	5.91e–004	2.79e–001	1.42e–004	1.41e+000	8.08e–004	2.41e–001
Deriv2_ex2	0.0e+000	8.33e–006	2.81e–002	8.33e–006	2.81e–002	8.49e–006	2.87e–002
Deriv2_ex2	1.0e–005	8.49e–006	4.40e–002	1.84e–005	5.98e–002	8.49e–006	4.40e–002
Deriv2_ex2	1.0e–003	1.31e–004	2.19e–001	9.11e–005	3.00e–001	1.70e–004	1.88e–001
Deriv2_ex2	1.0e–002	5.91e–004	2.88e–001	9.11e–005	2.75e+000	9.76e–004	2.31e–001
Deriv2_ex3	0.0e+000	9.58e–006	3.91e–004	9.58e–006	3.91e–004	8.49e–006	4.09e–004
Deriv2_ex3	1.0e–005	1.05e–004	3.13e–003	1.05e–004	3.13e–003	6.60e–005	4.85e–003
Deriv2_ex3	1.0e–003	4.04e–003	3.18e–002	2.53e–002	1.19e–001	6.38e–004	3.09e–002
Deriv2_ex3	1.0e–002	1.12e–002	7.55e–002	2.53e–002	1.20e–001	2.51e–003	4.92e–002

Table 9. Efficiency values on the different test problems: $Ef_{\min}$ is the minimum efficiency value; Ef_{mean} is the mean efficiency value; *Opt*% is the percentage of tests, where each method reached the maximum efficiency value, and *Fail*% is percentage of cases with a relative error greater than 10

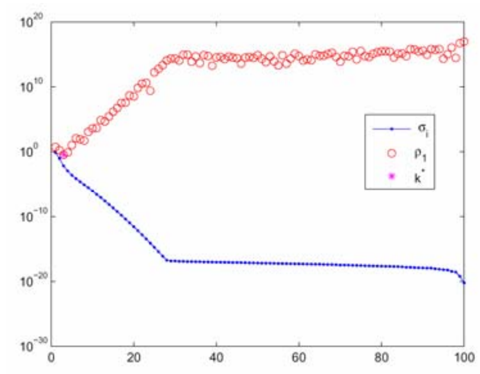
Test	**Method**	$Ef_{\min}$	Ef_{mean}	*Opt*%	*Fail*%
$G1$	**MinMax_tsvd**	3.54e-009	0.80	21	0
	GCV_tsvd	2.22e-020	0.59	23	27
	MinMax_tikh	7.21e-004	0.80	19	0
	GCV_tikh	5.34e-008	0.69	19	13
$G2$	**MinMax_tsvd**	1.4e-001	0.72	19	0
	GCV_tsvd	5.48e-011	0.55	19	0
	MinMax_tikh	1.72e-002	0.57	13	0
	GCV_tikh	1.68e-002	0.70	56	0



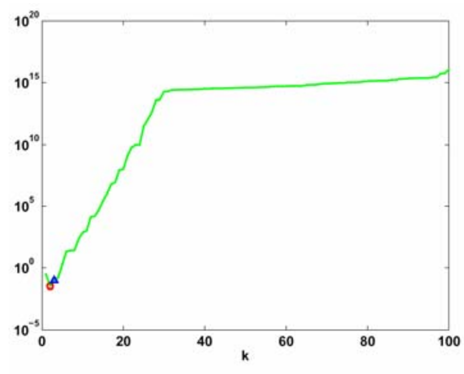
(a)



(b)



(c)



(d)

Figure 2. Foxgood test problem (a), (c) ρ_1 succession (red), ρ_2 succession (green); (b) (d) Relative error: red o (optimal solution), blue triangle (TSVD MinMax solution); (a)(b) noiseless case; (c)(d) high noise = 1.*e*–2.

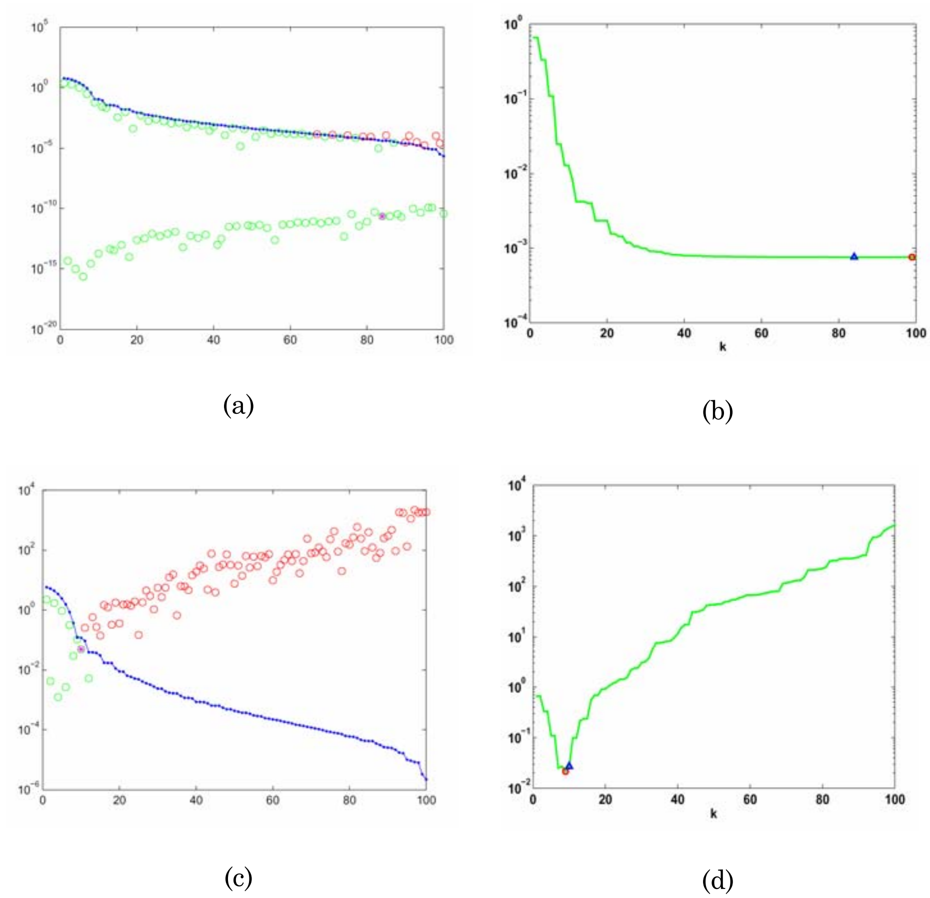
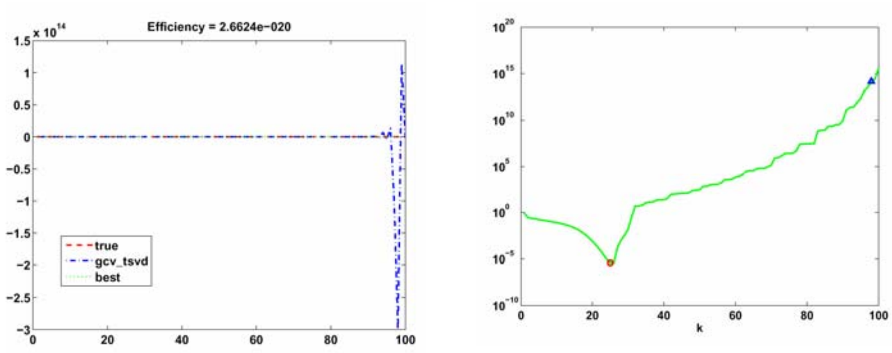
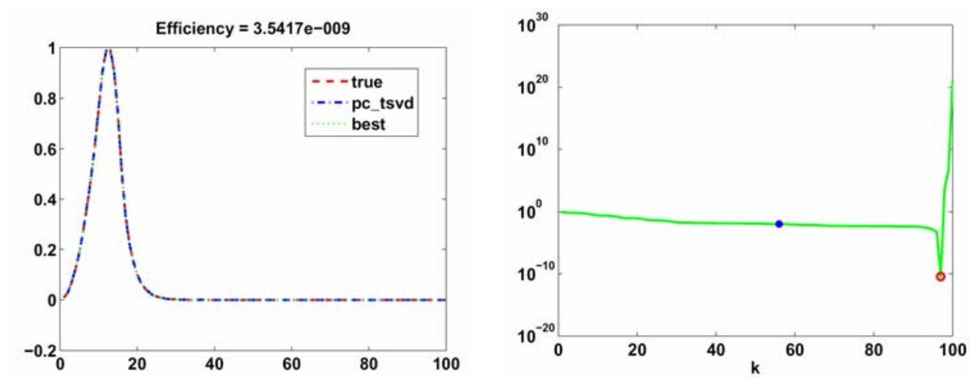


Figure 3. Phillips test problem (a), (c) ρ_1 succession (red), (ρ_2 succession (green) is empty); (b) (d) Relative error: red o (optimal solution), blue triangle (TSVD MinMax solution); (a)(b) noiseless case; (c)(d) high noise = 1.e–2.



(a) (b)

Figure 4. Worse efficiency values for GCV method: (a) Solution of ilaplace_ex1 test problem, no noise (b) Error curve for different values of the regularization parameters. Red dot: optimal parameter. Blue triangle: solution computed by tsvd with GCV parameter.



(a) (b)

Figure 5. Worse efficiency values for MinMax Rule: (a) Solution of heat test problem, no noise. (b) Error curve for different values of the regularization parameters. Red dot: optimal parameter. Blue triangle: solution computed by tsvd with regularization parameter given by MinMax Rule.

■